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Low-frequency approximation for high-intensity Compton scattering

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Abstract. An approximate expression is derived for the amplitude for the spontaneous emission of a photon by a charged spin-$\frac{1}{2}$ fermion as it is being deflected in an external field. The fermion is coupled to a spin-zero boson field through an interaction which is local and renormalisable. The external field is in the form of an intense pulse of radiation having a well defined direction of propagation. The essential restriction on the field is that it be slowly varying; its spectral composition is otherwise arbitrary. The transition amplitude obtained here depends only on the charge, mass and anomalous magnetic moment of the fermion. This result may be thought of as providing an external-field version of the low-energy theorem for ordinary Compton scattering which has been known for some years.

1. Introduction

Due to the advent of high-intensity laser sources there has been a fair amount of interest shown recently in theoretical studies of quantum electrodynamic processes which take place in the presence of an intense external electromagnetic field. (For a review, see Mitter 1975.) A typical process of interest, one with which we shall be concerned exclusively here, is the external-field version of Compton scattering: a charged particle is deflected in the laser field and emits a secondary (non-laser) quantum. The cross section for this process has been derived for the case where the charged particle is an electron, the interaction with the vacuum field being treated in lowest order (Brown and Kibble 1964, Narozhnyi *et al* 1965). Under what other circumstances can one still obtain reasonably simple expressions for the transition amplitude for such a process? In examining this question it is helpful to keep in mind the soft-photon approximation obtained some time ago by Low (1954) and by Gell-Mann and Goldberger (1954) for ordinary Compton scattering of systems of spin $\frac{1}{2}$. It was shown that the first two terms in an expansion of the amplitude in powers of the photon frequency could be expressed explicitly in terms of the charge, mass and magnetic moment of the scatterer. This suggests that similar simplifications may arise in the analysis of high-intensity Compton scattering and indeed this is borne out by the results obtained below.

Let us define in more explicit terms the model to be studied here. The scatterer is a charged spin-$\frac{1}{2}$ fermion locally coupled to spin-zero bosons. We assume the existence of a renormalised perturbation expansion in powers of this coupling and work to all orders in the expansion. Radiative corrections involving virtual photons are ignored here since we wish to avoid certain complications (infrared effects which are not central to our present concerns) associated with the zero mass of the photon. The external field is taken to be sufficiently intense so that it may be treated classically. The interaction of the spin-$\frac{1}{2}$ fermion with the external field in initial and final states is then accounted for by introducing solutions of the Dirac equation containing, in addition to the minimal coupling with the field, a phenomenological Pauli term which accounts for the effect of the anomalous magnetic moment. As discussed by Becker and Mitter (1974), and earlier in a more formal manner by Klein (1955), the assumption that the interaction with the external field can be characterised by the charge and static magnetic moment of the particle is a reasonable one for a low-frequency field. We also expect that the spontaneous emission vertex of the non-laser photon can be specified in terms of the physical values of the charge and magnetic moment of the scattered particle. This is verified in § 4, below. In doing so we make use of techniques developed earlier (Rosenberg 1982) in connection with the analysis of boson–fermion scattering in a slowly varying external field. Thus the characteristic and significant feature common to a class of low-frequency theorems, namely, that an amplitude for one process is determined by parameters which are in principle measurable by other experiments, is retained in the present version.

2. Formulation

We consider a spin-$\frac{1}{2}$ particle (a proton, say) of charge e, mass m and anomalous magnetic moment μ_A propagating in an external plane-wave field. The field is assumed to be slowly varying but no other assumption is made concerning its spectral composition and polarisation properties. In general the field will have an appreciable effect on the asymptotic motion of the particle so that a correct treatment of the particle–field interaction is required in the construction of the incoming and outgoing states. For the proton–field system these states are chosen as solutions of

$$[\gamma \cdot (\partial - \mathrm{i}eA) + m - \tfrac{1}{2}\mu_A \sigma_{\mu\nu} F^{\mu\nu}]\psi_p(x; A) = 0 \tag{2.1}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field tensor and $\sigma_{\mu\nu} = \frac{1}{2}\mathrm{i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$. The γ_μ are the usual Dirac matrices satisfying $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$, with $-g_{00} = g_{11} = g_{22} = g_{33} = 1$. (We use natural units with $\hbar = c = 1$.) The vector potential A_μ is taken to be a function of $u = -n \cdot x \equiv -n_\mu x^\mu$ with $n_\mu = (1, \boldsymbol{n})/\sqrt{2}$; here $\boldsymbol{n}$ is a unit vector specifying the unique propagation direction of the plane wave. The condition

$$\partial_\mu A^\mu = -\frac{\mathrm{d}}{\mathrm{d}u}(n \cdot A) = 0$$

is satisfied by requiring that $n \cdot A = 0$. The potential is assumed to vanish for $|u| > u_0$. It is then possible to construct linear combinations of (modified) plane-wave solutions of equation (2.1) representing localised wave packets which spend only a finite amount of time in the field (Neville and Rohrlich 1971).

Solutions $\psi_p^{(\pm)}(x; A)$ of equation (2.1) are required which satisfy the correct boundary conditions: $\psi_p^{(+)}$ reduces to the free plane wave $\exp(\mathrm{i}p \cdot x)\psi(p)$ for $u < -u_0$, as does $\psi_p^{(-)}$ for $u > u_0$. Here $\psi(p)$ is the free spinor satisfying $(\mathrm{i}\gamma \cdot p + m)\psi(p) = 0$ for $p^2 + m^2 = 0$. As demonstrated in detail in the papers of Becker and Mitter (1974) and Becker (1975) the solutions take the form

$$\psi_p^{(\pm)}(x; A) = \exp(\mathrm{i}p \cdot x) \exp(-\mathrm{i}S_p^{(\pm)}(u))\chi_p^{(\pm)}(u) \tag{2.2a}$$

with

$$S_p^{(+)}(u)=\int_{-u_0}^{u} I_p(\bar{u})\,\mathrm{d}\bar{u} \tag{2.2b}$$

$$S_p^{(-)}(u)=-\int_{u}^{u_0} I_p(\bar{u})\,\mathrm{d}\bar{u} \tag{2.2c}$$

$$I_p(u)=(2n\cdot p)^{-1}[2ep\cdot A(u)-e^2A^2(u)]. \tag{2.2d}$$

We write

$$\chi_p^{(\pm)}(u)=J_p^{(\pm)}(u)\Big(1+\frac{e}{2n\cdot p}\gamma\cdot n\gamma\cdot A\Big)\psi(p). \tag{2.3}$$

For $\mu_A=0$ we have $J_p^{(\pm)}(u)=1$, corresponding to the well known solution of Volkov (1935). Construction of $J_p^{(\pm)}(u)$ for the more general case has been discussed by Becker and Mitter (1974) and by Becker (1975) who show how the problem may be reduced to one involving a set of coupled first-order differential equations (which may be solved numerically). We shall not concern ourselves here with the general problem of solving these equations (explicit solutions have been given by Becker and Mitter for the special cases of a linearly polarised wave and a circularly polarised monochromatic wave of infinite extent) but shall proceed here under the assumption that solutions are available in some approximation. Fortunately, a relation satisfied by the functions $\chi_p^{(\pm)}$ which, as we shall see in § 4, is crucial in the development of the low-frequency approximation, can be established directly using known general properties of the solution (see equation (3.4) of the paper by Becker (1975)). That relation is

$$[\mathrm{i}\gamma\cdot p(u)+m]\chi_p^{(\pm)}(u)=0 \tag{2.4}$$

where

$$p(u)=p-eA(u)+nI_p(u). \tag{2.5}$$

It is interesting, with regard to physical interpretation, to observe that $p(u)$ may be identified as the classically determined momentum for the charged particle in the field (Brown and Kibble 1964). It is of course not $\psi_p^{(-)}$ which appears in the matrix element but $\bar{\psi}_p^{(-)}=\bar{\chi}_p^{(-)}\exp(-\mathrm{i}p\cdot x)\exp\mathrm{i}S_p^{(-)}$. Here $\bar{\chi}_p^{(-)}$ is the adjoint spinor, constructed from $\chi_p^{(-)}$ in the usual way (Jauch and Rohrlich 1976). Note that $S_p^{(+)}$ and $S_p^{(-)}$ differ only by a constant. It will be convenient to ignore this difference (which merely redefines the phase of the transition matrix element) and take

$$S_p^{(\pm)}\equiv S_p=\int_{-u_0}^{u} I_p(\bar{u})\,\mathrm{d}\bar{u}.$$

We consider a transition in which a proton, before it enters the field, is in a state with momentum p and spin index s. It leaves the field with momentum p' and spin s', having emitted a non-laser photon of momentum q' and polarisation ε'. The invariant amplitude for the transition may be written (with spin and polarisation indices suppressed) in the form

$$\tilde{\mathcal{T}}(p',q';p;A)=\int \mathrm{d}^4x'\,\mathrm{d}^4y'\,\mathrm{d}^4x\,\bar{\psi}_{p'}^{(-)}(x';A)\exp(-\mathrm{i}q'\cdot y')\mathcal{T}(x',y';x;A)\psi_p^{(+)}(x;A). \tag{2.6}$$

The form factor $\mathcal{T}(x', y'; x; A)$ represents the collection of Feynman matrix elements in configuration space which constitutes the spontaneous emission vertex. The modified Feynman rules, taking into account the presence of the external field, have been listed by Mitter (1975). In the absence of the field we have $\mathcal{T}(x', y'; x; A) \to \mathcal{T}(x'-y'; x-y')$ with

$$\int d^4x'\, d^4y'\, d^4x \exp i(-p'\cdot x' - q'\cdot y' + p\cdot x)\mathcal{T}(x'-y'; x-y')$$
$$= (2\pi)^4\delta^4(p'+q'-p)e\varepsilon'^{\mu}\Gamma_\mu(p', p); \qquad (2.7)$$

here Γ_μ is the field-free vertex part in momentum space. A low-frequency approximation for Γ_μ, which is consistent with the use of the modified Dirac equation (2.1) to describe the interaction of the proton with the *external* low-frequency field, is given by

$$e\Gamma_\mu^{\text{static}}(p', p) = e\gamma_\mu + \mu_{\text{A}}\sigma_{\mu\nu}(p'-p)^\nu \qquad (2.8)$$

where e and μ_{A} represent the physical charge and anomalous magnetic moment, respectively, of the proton.

3. Static approximation for the spontaneous emission vertex

Here we neglect the effect of the external field on the structure of the spontaneous emission vertex and, furthermore, adopt the static approximation (2.8) for the vertex function. We return, in § 4, to an analysis of the validity of this approximation.

We begin with equation (2.6), but with the form factor $\mathcal{T}(x', y'; x; A)$ replaced by its field-free value. To facilitate passage to momentum space we introduce the identity

$$\bar{\psi}_{p'}^{(-)}(x'; A)\mathcal{T}(x'-y'; x-y')\psi_p^{(+)}(x; A) = \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}d\bar{u}'\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}d\bar{u}$$
$$\times \exp i[-\omega'(\bar{u}'-u') + \omega(\bar{u}-u)]\bar{\chi}_{p'}^{(-)}(\bar{u}')\exp(iS_{p'}(\bar{u}'))\exp(-ip'\cdot x')$$
$$\times \mathcal{T}(x'-y'; x-y')\exp(ip\cdot x)\exp(-iS_p(\bar{u}))\chi_p^{(+)}(\bar{u}). \qquad (3.1)$$

(This identity is immediately verified by first performing the integrations over ω and ω' using the integral representation of the δ function.) The integrations over x', y' and x in the approximate version of equation (2.6) may now be carried out as in equation (2.7), but with Γ_μ replaced by $\Gamma_\mu^{\text{static}}$. The result is

$$\tilde{\mathcal{T}}^{\text{static}}(p', q'; p, A) = \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}d\bar{u}'\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}d\bar{u}$$
$$\times (2\pi)^4\delta^4[p'+q'-p+(\omega'-\omega)n]\exp i[\omega\bar{u} - S_p(\bar{u}) - \omega'\bar{u}' + S_{p'}(\bar{u}')]$$
$$\times e\varepsilon'^{\mu}\bar{\chi}_{p'}^{(-)}(\bar{u}')\Gamma_\mu^{\text{static}}(p'+\omega' n, p+\omega n)\chi_p^{(+)}(\bar{u}). \qquad (3.2)$$

Further simplification can be achieved by making use of the light-like coordinates (Mitter 1975). These are defined in terms of the four basis vectors n, $\hat{n} = (1, -\boldsymbol{n})/\sqrt{2}$ and the real, orthogonal polarisation vectors ε_1 and ε_2, each orthogonal to n and $\hat{n}$. In this basis the components of an arbitrary vector V are $V_1 = \varepsilon_1\cdot V$, $V_2 = \varepsilon_2\cdot V$,

$V_v = -n \cdot V$ and $V_u = -\hat{n} \cdot V$. Writing the δ function in equation (3.2) as

$$\delta(p_1' + q_1' - p_1)\delta(p_2' + q_2' - p_2)\delta(p_v' + q_v' - p_v)\delta(p_u' + q_u' - p_u + \omega' - \omega)$$

we may perform the integration over ω' and set

$$\omega' = \omega - (p_u' + q_u' - p_u) \tag{3.3}$$

elsewhere in the integrand. Since the static vertex function depends only on q' the integrations over ω and $\bar{u}'$ are easily carried out and we find

$$\begin{aligned}\tilde{\mathcal{T}}^{\text{static}}(p', q'; p; A) = (2\pi)^3 \delta(p_1' + q_1' - p_1)\delta(p_2' + q_2' - p_2)\delta(p_v' + q_v' - p_v)\\ \times \int_{-\infty}^{\infty} d\bar{u} \exp \mathrm{i}\,[S_{p'}(\bar{u}) - S_p(\bar{u}) + (p_u' + q_u' - p_u)\bar{u}]\\ \times \varepsilon'^{\mu} \bar{\chi}_{p'}^{(-)}(\bar{u})(e\gamma_\mu - \mu_{\mathrm{A}}\sigma_{\mu\nu}q'^{\nu})\chi_p^{(+)}(\bar{u}).\end{aligned} \tag{3.4}$$

Further reduction of this expression requires a specification of the form of the vector potential $A(u)$. Brown and Kibble (1964) assumed a monochromatic plane wave of infinite extent and studied the Compton scattering of an electron, working to first order in the interaction of the electron with the vacuum field. The expression (3.4), when evaluated for the case of a monochromatic wave, differs from that arrived at by Brown and Kibble by the presence of terms involving μ_{A}. We shall not take the space here to study these extra terms in detail. We restrict ourselves to the observation that an examination of the expression (3.4) in the weak-field limit provides us with a partial check on its validity. To study this limit we may employ the approximate form for $\chi_p^{(\pm)}$, valid to first order in A, which was given earlier (Rosenberg 1982). It is then a simple matter to derive the amplitude for the process in which a single laser photon is absorbed and a non-laser photon is emitted. As expected, it takes the form one would obtain from second-order perturbation theory with emission and absorption vertices given by equation (2.8).

4. Analysis of spontaneous emission vertex

We return now to the expression (2.6) for a more careful analysis. We first observe that our earlier assumption that the effect of the external field on the form factor $\mathcal{T}(x', y'; x; A)$ may be neglected entirely is suspect since it leads to an approximation which fails to satisfy gauge invariance. That is, with $A_\mu(u) \to A_\mu(u) + \partial_\mu\lambda(u) = A_\mu(u) - n_\mu\, d\lambda/du$ we have

$$\bar{\psi}_{p'}^{(-)}(x'; A)\psi_p^{(+)}(x; A) \to \exp[\mathrm{i}(\lambda(u) - \lambda(u'))]\bar{\psi}_{p'}^{(-)}(x'; A)\psi_p^{(+)}(x; A)$$

so that the product $\bar{\psi}_{p'}^{(-)}(x'; A)\mathcal{T}(x' - y'; x - y)\psi_p^{(+)}(x; A)$ is not invariant. The final version (3.4) *is* gauge invariant. This has come about from an improper treatment of the off-mass-shell contributions to the vertex function which just compensates for the original violation of gauge invariance.

An improved approximation for the form factor $\mathcal{T}(x', y'; x; A)$ which preserves gauge invariance can be obtained in a very simple form for the case of a slowly varying external field. At this point we shall follow very closely the arguments presented earlier (Rosenberg 1982) in an analysis of boson–fermion scattering in an external field. (We shall refer to that earlier work as I.) The approximate kernel is taken to be of the form

$$\mathscr{T}(x', y'; x; A) \simeq \exp[ie\Lambda(x', x)]\mathscr{T}(x'-y, x-y') \tag{4.1}$$

with

$$\Lambda(x', x) = (x'-x) \cdot \int_0^1 d\eta\, A[u + \eta(u'-u)]. \tag{4.2}$$

More generally (Schwinger 1951) $\Lambda(x', x)$ can be represented by a path-independent integral, reducing to (4.2) when the path is taken to be a straight line. The justification for the approximation (4.1) has been given in I for the analogous case of the 4-point function. To summarise that argument briefly, we consider the representation of $\mathscr{T}(x', y'; x; A)$ as a collection of Feynman diagrams in configuration space. As shown in I the charged fermion propagator in the presence of the slowly varying external field may be approximated by

$$G_f(x', x; A) \simeq \exp[ie\Lambda(x', x)]S_c(x'-x; m) \tag{4.3}$$

where S_c is the field-free causal fermion propagator. As a result of the local interaction with the spin-zero boson field charge may be passed from a fermion line to a boson line. Using an argument similar to that used in arriving at equation (4.3) we obtain an approximation for the boson propagator of the form

$$G_b(x', x; A) \simeq \exp[ie\Lambda(x', x)]\Delta_c(x'-x; \mu^2) \tag{4.4}$$

where $\Delta_c(x'-x; \mu^2)$ is the free propagator for a spin-zero particle of mass μ. As a consequence of the approximations (4.3) and (4.4), along with the path-independence property of the line integral, a charged particle line passing continuously through the diagram picks up an overall phase $e\Lambda(x', x)$, the *same* phase for each diagram. Since closed charged particle loops have no additional phase factors associated with them the only effect of the field in this approximation is to introduce the phase factor shown in equation (4.1). When this equation is combined with equation (2.6) we obtain the gauge invariant approximation

$$\tilde{\mathscr{T}}(p', q'; p; A) \simeq \int d^4x'\, d^4y'\, d^4x\, \bar{\psi}_{p'}^{(-)}(x'; A) \times \exp(-iq' \cdot y') \exp[ie\Lambda(x', x)]\mathscr{T}(x'-y', x-y')\psi_p^{(+)}(x; A) \tag{4.5}$$

for the transition amplitude.

Further analysis is facilitated by the introduction of the identity (generalising that shown in equation (3.1) above)

$$\begin{aligned}\bar{\psi}_{p'}^{(-)}(x'; A) &\exp[ie\Lambda(x', x)]\mathscr{T}(x'-y'; x-y')\psi_p^{(+)}(x; A)\\ &= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} d\bar{u}' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} d\bar{u}\, \bar{\chi}_{p'}^{(-)}(\bar{u}')\mathscr{T}(x'-y'; x-y')\chi_p^{(+)}(\bar{u})\\ &\quad \times \exp i[\omega\bar{u} - S_p(\bar{u}) + Q_p(\omega, \bar{u}', \bar{u}) \cdot x - \omega'\bar{u}'\\ &\quad + S_{p'}(\bar{u}') - Q_{p'}(\omega', \bar{u}', \bar{u}) \cdot x']\end{aligned} \tag{4.6}$$

with

$$Q_p(\omega, \bar{u}', \bar{u}) = p + \omega n - e\int_0^1 d\eta\, A[\bar{u} + \eta(\bar{u}'-\bar{u})]. \tag{4.7}$$

The transition amplitude (2.6) may then be put in the form (3.2), with $\Gamma_\mu^{\text{static}}$ replaced by $\Gamma_\mu[Q_{p'}(\omega', \bar{u}', \bar{u}), Q_p(\omega, \bar{u}', \bar{u})]$.

It will be convenient in the following to work in a particular gauge (the radiation gauge), in which $A_u = 0$. If, in accordance with the assumption that the field is slowly varying, we ignore the variation of A in equation (4.7) we may write

$$Q_p(\omega, \bar{u}', \bar{u}) \simeq Q_p(\omega, \bar{u}) \equiv p + \omega n - eA(\bar{u}) \tag{4.8a}$$

$$Q_{p'}(\omega', \bar{u}', \bar{u}) \simeq Q_{p'}(\omega', \bar{u}') \equiv p' + \omega' n - eA(\bar{u}'). \tag{4.8b}$$

Introduction of the light-like coordinates allows us to reduce the expression for the transition amplitude to

$$\begin{aligned}\tilde{\mathcal{T}}(p', q'; p; A) \simeq (2\pi)^3 \delta(p_1' + q_1' - p_1)\delta(p_2' + q_2' - p_2)\delta(p_v' + q_v' - p_v) \\ \times \int_{-\infty}^{\infty} \mathrm{d}\bar{u} \int_{-\infty}^{\infty} \mathrm{d}\bar{u}' \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \exp \mathrm{i}[\omega(\bar{u} - \bar{u}') + S_{p'}(\bar{u}') - S_p(\bar{u}) \\ + (p_u' + q_u' - p_u)\bar{u}'] e\varepsilon'^{\mu} \bar{\chi}_{p'}^{(-)}(\bar{u}') \Gamma_\mu[Q_{p'}(\omega', \bar{u}'), Q_p(\omega, \bar{u})] \chi_p^{(+)}(\bar{u})\end{aligned} \tag{4.9}$$

with ω' given by equation (3.3). A measure of the degree to which the vertex function is off the mass shell is provided by the scalar variables

$$\xi = Q_p^2(\omega, \bar{u}) + m^2 = 2n \cdot p[\omega - I_p(\bar{u})] \tag{4.10a}$$

$$\xi' = Q_{p'}^2(\omega', \bar{u}') + m^2 = 2n \cdot p'[\omega' - I_{p'}(\bar{u}')]. \tag{4.10b}$$

The off-shell variables ξ and ξ' may be set equal to zero under a wide range of conditions consistent with the basic assumption of a slowly varying field. Thus, for S_p of order unity and I_p taken to be a quantity of first order (the intermediate-coupling regime) a Taylor series expansion of the vertex function about $\xi = \xi' = 0$ may be introduced and, using an integration by parts procedure, the correction terms of first order in ξ and ξ' may be shown to vanish (see I for the details of this argument). The series expansion technique fails in the strong-coupling regime, characterised by the condition $S_p(\bar{u}) \gg 1$ inside the laser pulse. Instead we may apply a stationary phase argument, based on the rapid variation of the exponential in equation (4.9) as a function of $\bar{u}$ for $S_p(\bar{u}) \gg 1$. The stationary phase condition is $\omega = I_p(\bar{u})$. After replacing ω by this value in the argument of the vertex function in equation (4.9) the integration over ω may be carried out. This introduces a δ function in $(\bar{u}' - \bar{u})$ so that the $\bar{u}'$ integration may be performed as well. A second application of the stationary phase argument leads to the condition

$$p_u' + q_u' - p_u = I_p(\bar{u}) - I_{p'}(\bar{u}). \tag{4.11}$$

Setting $\omega = I_p(\bar{u})$ in equation (4.8*a*) gives

$$Q_p(\omega, \bar{u}) \to p + I_p(\bar{u})n - eA(\bar{u}) \equiv p(\bar{u}) \tag{4.12}$$

where $p(\bar{u})$ is the classically determined momentum introduced earlier in equation (2.5). Similarly, from equations (4.8*b*) and (4.11) we find that

$$Q_{p'}(\omega', \bar{u}) \to p' + I_{p'}(\bar{u})n - eA(\bar{u}) \equiv p'(\bar{u}). \tag{4.13}$$

Equation (4.9) now becomes

$$\tilde{\mathcal{T}}(p', q'; p; A) \simeq (2\pi)^3 \delta(p_1' + q_1' - p_1)\delta(p_2' + q_2' - p_2)\delta(p_v' + q_v' - p_v)$$
$$\times \int_{-\infty}^{\infty} d\bar{u} \exp i [S_{p'}(\bar{u}) - S_p(\bar{u}) + (p_u' + q_u' - p_u)\bar{u}]$$
$$\times e\varepsilon'^{\mu} \bar{\chi}_{p'}^{(-)}(\bar{u})\Gamma_{\mu}[p'(\bar{u}), p(\bar{u})]\chi_p^{(+)}(\bar{u}). \tag{4.14}$$

The identical result is obtained by analysing the intermediate-coupling case along the lines indicated in I.

The vertex function in equation (4.14) is on the mass shell since $p^2(\bar{u}) + m^2 = p'^2(\bar{u}) + m^2 = 0$. Furthermore, by virtue of the conditions (2.4) the vertex may be represented in terms of two functions (the electromagnetic form factors) of the scalar variable $[p(\bar{u}) - p'(\bar{u})]^2$ (Schweber 1961). It follows from equation (4.11), and the presence of the δ functions in equation (4.9), that at the point of stationary phase $p(\bar{u}) - p'(\bar{u}) = q'$. Since $q'^2 = 0$ each form factor is evaluated for zero value of its argument. The result of these considerations is that the vertex function in equation (4.14) is to be replaced by its static limit, equation (2.8), and that equation (3.4) for the transition amplitude is recovered.

5. Summary

It has been shown that the assumption of the static approximation (2.8) for the spontaneous emission vertex function is valid under less restrictive circumstances than might have been expected *a priori.* The following remarks may be made in this connection. (i) Lowest-order perturbation theory for the fermion–boson interaction is not assumed. The parameters e and μ_A are renormalised quantities. The result is then analogous to that obtained by Low (1954) and by Gell-Mann and Goldberger (1954) who showed that the amplitude for ordinary Compton scattering of low-energy photons is given by second-order perturbation theory with vertices determined by the physical values of the static charge and magnetic moment of the scatterer. (ii) The effect of the external field on the vertex function has not been ignored entirely. To do so would violate gauge invariance and, for strong enough fields, would certainly be inappropriate. Rather, for fields which are slowly varying but not necessarily weak, the effect of the field on the vertex function is accounted for in an approximate way by the introduction of the phase factor shown in equation (4.1). The field has its primary influence on the incoming and outgoing states and that is treated exactly. (iii) The deviation of the vertex function off the mass shell need not be neglected at the outset. Off-shell effects are retained, but are seen to cancel under a wide range of conditions (characterised here as intermediate- and strong-coupling regimes) consistent with the underlying assumption of a slowly varying field. (iv) If the field is strong enough (strong-coupling case) many laser photons can be absorbed in the scattering process, with the spontaneously emitted photon carrying off an appreciable amount of energy. Nevertheless, the vertex function is to be evaluated on shell, as we have seen, so that even in this case only the static charge and magnetic moment of the scattered fermion enter into the calculation. (v) The choice of incoming and outgoing states as solutions of equation (2.1) is based on the assumption that the particle–field interaction is determined, asymptotically, by the static charge and magnetic moment of the particle. It would then appear to be required, for consistency, to adopt the static approximation for the spontaneous emission vertex. The present discussion, in so far as it verifies the validity of the static approximation, argues in support of the self-consistency of the calculation.

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